ON SET-TALK AND THE PHILOSOPHY OF LANGUAGE: A CASE STUDY

1. *Introduction*

It is quite commonly believed that the central, if not the only, question in the philosophy of mathematics is an ontological one: to be or not to be. Sets occupy a large place here; so, when the "linguistic turn" is applied to the ontological question, philosophers are once and again heard to ask what a mathematician who engages in set-talk is really talking about. Everyone will readily admit that the very formulation of this question presupposes the primacy of ontology. Now, ontology, from the time it first appeared in the West till our own time, has always had a polemical character. Parmenides didn't believe there was anything but the unchangeable, unmovable One, everything else being only illusion, and Aristotle didn't believe there were any Ideas. Following in the steps of those giants, our modern ontologists don't believe there are any sets. On the other hand, there is set-talk, and a very useful talk it is, too. No one in his right mind can deny that. In the old times, the situation was not very different. Talk of movement and change was useful, even Parmenides had to admit as much. His solution to the difficulty: that sort of talk appertains to the field of opinion, not truth. (There are philosophers today who would like to solve the problem of set-talk in very much the same way; but we are not dealing with them here.) Aristotle's solution was a bit more sophisticated. He saw that talk of Ideas was, among other things, useful because it provided a rationale for definitions, which were necessary for science. His task was then to construct a different theory of definition that could do without Ideas. And in all the other fields of knowledge where it seemed necessary to talk of Ideas (e.g. in ethics), he tried to build up a different sort of talk. That is also, mutatis mutandis, the strategy that modern ontologists like most. They say, "Well, mathematicians talk of sets, but there aren't any sets; so they must really be trying to talk of things that are not sets, but somehow they don't get to it. Therefore, we must find a way to reduce their set-talk to another sort of talk that is not ostensibly of sets, but of objects of which we know that they exist."[1]

Now, the first point to urge here, from a linguistic point of view, would be that mathematicians are not talking of "sets". Indeed, the most cursory examination of mathematical texts will show that the only writings where a solitary set-word (or almost) is used happen to be set-theoretical in nature, e.g. the very popular textbook by Erich Kamke (1971)—and even there you will find, besides the word *Menge* (i.e. *set* in the English translation), the words *Komplex* (§ 13), *Kette* (§ 21), *Klasse* (§ 25), *Folge* (§ 32), among many others. And if we consider another aspect of set-talk, namely composition, we have *Untermenge, Teilmenge, Obermenge, Restmenge, Nullmenge, Vereinigungsmenge, Potenzmenge, Maximalmenge*, and so on and so forth. Or take adjectives: *abzählbare Mengen, wohlgeordnete Mengen, dichte Mengen, stetige Mengen*, etc. How can a philosopher speak as though Kamke were just talking of "sets"? And if not even a set-theoretician can be correctly described as talking of "sets", the case of an algebraist is much more complicated: if a philosopher wants to get rid of van der Waerden's (1971) set-talk, he should show us how to get rid, not only of the word *Menge*, but also and especially of the words *Gruppe, Ring, Körper, Ideal, Vektorraum, Tensorraum, Galois-Feld*, and of all the constructions based on them. To reply that the "objects" in question are all "sets", because they are defined as "sets" is to miss the point completely. But the usual philosopher has got a vested interest in believing that set-talk is simply talk of "sets", because, as I said, he is an ontologist, doesn't believe there are any sets, and wants to get rid of set-talk. His first move is always to simplify set-talk in such a way that any working mathematician, as the word goes, would scarcely recognize it as his own should he read philosophical discussions (which he doesn't, of course, otherwise he wouldn't be a working mathematician). The question is: are philosophers missing something given this first move?

On the other hand, I cannot as a linguist accept the primacy of ontology. I think that the only linguistically defensible position as to "objects" is to insist that everybody—including ontologists—talks in such a way that "objects" become constituted by that very same talk; that language is not an inventory of labels to name things with, and these things all the time out there, in the world, waiting patiently for a man or woman to take care of them; that "objects" have to be linguistically "constructed" ("apprehended", "individuated") by means of very complex linguistic techniques, so much so that for a linguist there is not, and cannot be, any concept of an "object" or "individual" that is independent of language.[2] That leads us to a sharp distinction, both in method and in theory: the philosopher wants to reduce set-talk, the linguist wants to understand it; the philosopher wants to prescribe people how they should talk, the linguist wants to describe how they actually talk; the philosopher believes there aren't any sets, the linguist doesn't see the point of believing one thing or the other, only he wants to point out the great complexity of set-talk; the philosopher starts by ignoring this complexity, the linguist by plunging directly into it; the philosopher asks what the mathematician is talking about, the linguist asks why the mathematician talks in such and such a manner. It would be difficult to find two persons who think and act more differently.

2. *A case in point*

I am fully aware that readers who believe that the ontological question is the question won't be convinced by what I have to say. But I want to say it all the same: ontology is a very bad way to ask questions—a very primitive and unfruitful way—at least if you have intellectual curiosity. There are lots of things, especially things connected with language, but also many connected with knowledge, thinking, creativity, and even with history, that can be asked and answered—things of much greater interest and much greater consequence than all ontological odds and ends. For, when all is said and done, what have we learnt from the ontological pundits that has really advanced our understanding of mathematics, either of theory, of language or of thinking?[3]

Take, for example, Dedekind (1872). The story of this brief but consequential treatise is told by the author in a prologue: for certain very important theorems of calculus there was in his time no rigorous proof—in fact, there hadn't been any for centuries. Instead, mathematicians used to appeal to geometrical intuition (as they still do today, by the way). We believe we know by geometrical intuition that space is continuous, and since calculus deals especially with continuous magnitudes, that appeal seems particularly apt, and, as Dedekind remarks, "from a didactical viewpoint extraordinarily useful, indeed unavoidable in case you don't want to lose too much time" (1872: 4). But naturalness and expediency are no substitutes of rigour, at least not indefinitely. In particular, geometrical intuition is no susbtitute of arithmetic proof: it shouldn't be forgotten that the real numbers are the domain of calculus, and if that domain is supposed to possess a certain property (in our case, continuity), then this has to be proved arithmetically, for arithmetic is the science of numbers.

That proof was Dedekind's main purpose. To arrive to it, to develop its consequences, and thus to lay down the foundations of calculus and pave the way for a new exposition of this discipline—this was a feat that costed Dedekind many years of hard thinking, of "productive" thinking in the sense of Wertheimer (1945). Now, the linguistic point of it all is that such productive thinking was made possible by some set-words and set-expressions which Dedekind either devised for himself or took from others for that very purpose. To put it in a nutshell, Dedekind creates a new "object", namely the domain of all real numbers, in order to predicate continuity of it. Continuity, and also well-ordering and two-sided infinity, which are its preconditions, constitute together a "limited universe of discourse", within which there is such a thing as the domain of all real numbers, because only such a thing can be said to possess those properties. Incidentally, one of the results of my analysis is that a predicate such as *(is) continuous* constitutes only the culmination of a whole "continuum of predication" (in Seiler's sense of the word[4]). This continuum begins, in our case, with predicates that clearly apply to individual numbers (or almost); it goes further through all sorts of increasingly "setty" predicates; and, by easy steps, you arrive to so "deep" and "embracing" properties that you cannot turn back any more—where even the most careful of reductionists must admit to an "irreducible core" of set-talk.[5] Moreover, I strongly suspect that the famous "mother structures" of the Bourbaki group illustrate precisely this point: order is less "setty" than algebraic relations, and algebraic relations are less "setty" than topological properties.[6] On the other hand, the new "objects" themselves are not created all of a sudden, but they are motivated by the appearance of the correlating predicates—and so they are not all "objective" in the same degree. As elsewhere in heaven or earth, things are much more subtle here than any ontology could ever dream.

3. *Some figures*

As a matter of fact, Dedekind uses nine set-words: *Reihe*, *Kette*, *System*, *Körper*, *Gebiet*, *Klasse*, *Stück*, *Schnitt* and *Intervall*. They are mentioned here in order of appearance, because my analysis of Dedekind's set-talk tries to respect the train of his thought. But I would like to add the word *Gerade* (or the equivalent expression *gerade Linie* and even *Linie*) to that list, because Dedekind himself considers a straight line—and space in general—very much like a set, namely a set of points.

How often does Dedekind use these ten set-words? His text is divided into seven sections (or *Paragraphen*) which attain a very accurate articulation of our author's train of thought. For future reference I give the headings of these sections:

(1) § 1. *Eigenschaften der rationalen Zahlen.*
§ 2. *Vergleichung der rationalen Zahlen mit den Punkten einer geraden Linie.*

§ 3. *Stetigkeit der geraden Linie.*
§ 4. *Schöpfung der irrationalen Zahlen.*
§ 5. *Stetigkeit des Gebietes der reellen Zahlen.*
§ 6. *Rechnungen mit reellen Zahlen.*
§ 7. *Infinitesimal-Analysis.*

Observe the underlined words. At these stage of the discussion, we can say that they indicate the "objects" apprehended: the rational numbers, the points of a straight line, the straight line, the irrational numbers, the domain of the real numbers, and (individual?) real numbers. So far, so good. Now observe the words that are not underlined. They indicate the predicates that are said of those "objects", the universe of discourse to which those "objects" appertain: the rational numbers are said to have properties, they are also said to be compared to the points of a straight line, the straight line is said to be continuous, the irrational numbers are said to be created, the domain of the real numbers is said—like the straight line—to be continuous, and (individual?) real numbers are said to be calculated with. As I suggested before, this is happening all the time in mathematical texts (and therefore also in Dedekind's text): you seem to be able to say some things of e.g. numbers and points, but unable to say other things of them. Here in particular, the word *Stetigkeit* is always and only accompanied by a set-word (either *Gebiet* or *Gerade*). And the other predicates seem to agree better with a plural NP with no set-word. What does this mean?

Now, if we examine the distribution of set-words in each section, we obtain the following frequencies (all occurrences have been taken into account, i.e. not only the full nouns, but also their substitutions by means of pronouns, other NPs and variables):

(2)

	§1	§2	§3	§4	§5	§6	§7	*Total*
Reihe	2	-	-	-	-	-	-	2
Kette	2	-	-	-	-	-	-	2
System	9	1	-	2	5	8	12	37
Körper	1	-	-	-	-	-	-	1
Gebiet	1	-	5	2	3	1	1	13
Klasse	12	10	4	75	36	17	-	154
Gerade	-	10	16	-	-	-	-	26
Stück	-	1	4	-	-	2	-	7
Schnitt	-	-	-	31	3	8	3	45
Intervall	-	-	-	-	-	14	-	14
	(27)	(22)	(29)	(110)	(47)	(50)	(16)	(301)

These figures show at least four things:

(a) the significance of the words *Reihe, Kette* and *Korper* is somehow restricted to §1, that of the word *Gerade* to §§2-3, and that of the word *Intervall* to §6;

(b) the word *Schnitt* becomes significant in the middle of the book (§4), but from that point on does never cease to be so;

(c) the word *Stück* behaves in a somewhat erratic way;

(d) the words *System, Gebiet* and *Klasse* are the most evenly distributed set-words in the text with only a non-occurrence relative to one section each, i.e. specifically the word *Gebiet* doesn't seem to be significant for §2, *System* for §3, and *Klasse* for §7.

Of course, there are other ways to arrange our 301 occurrences. To arrive at a very interesting one, consider the only plausible case of synonymy: the words *System* and *Gebiet* are quite often used to "refer to" either the set of rational numbers (case A) or the set of real numbers (case B):[7]

(3)

	A	*B*	*Total*	
System	15	6	21	[from 37]
Gebiet	6	3	9	[from 13]
	(21)	(9)	(30)	

This looks like redundancy, and redundancy is not very obviously related to productive thinking. However, a closer look reveals that there is, in Dedekind's set-talk, a sort of "switching" from *System* to *Gebiet* and back—a switching that is consistently motivated by the appearance of certain predicates, namely the above-mentioned "setty" predicates of well-ordering, two-sided infinity and continuity. These predicates are everywhere applied only to NPs whose head-noun is the word *Gebiet*, so much so that if we use this criterion, we find a much more interesting distribution:

(4)

	+	-
System	0	37
Gebiet	13	0

And what happens with the 16 occurrences of the word *System* that don't serve to "refer to" one of the above-mentioned sets? As a matter of fact, they are all cases where *System* is used to "refer to" non-specified subsets of one of them. This forces us to consider the fact that a very usual way to arrive to a set is by separating it from a given set. If we apply this criterion to our ten set-words, we arrive to the following chart:

(5)

	+	-
Reihe	0	2
Kette	0	2
System	16	21
Körper	0	1
Gebiet	0	13
Klasse	154	0
Gerade	0	26
Stück	7	0
Schnitt	45	0
Intervall	14	0
	(236)	(65)

In other words, with the sole exception of *System*, which seems to live a sort of double existence, all set-words in Dedekind's text belong to one of two classes: they either indicate an "extraction" from a given set or they don't.[8] The case of *Schnitt* is special in that this word does not "refer to" subsets, but to sets of subsets, as we shall see. A very interesting regularity here is this: any set-word can be accompanied or substituted by a variable if and only if it "refers to" a set that is "extracted" from a given set. We can sum up the discussion by distinguishing two cases with three subvariants each:

CASE I, when a set-word is used to refer to a specified set, namely either to the set of positive integers (Ia) or to the set of points of a straight line (Ib) or to any of the sets of rational or real numbers (Ic);

CASE II, when a set-word is not so used, in which case it can be accompanied/substituted by variables in referring to subsets (IIa) or to sets of subsets (IIb), or else not so accompanied/substituted (IIc).

The distribution of our 301 occurrences would be then as follows:

(6)

	Ia	*Ib*	*Ic*		*IIa*	*IIb*	*IIc*	
Reihe	2	-	-	(2)	-	-	-	(0)
Kette	2	-	-	(2)	-	-	-	(0)
System	-	-	21	(21)	16	-	-	(16)
Körper	-	-	-	(0)	-	-	1	(1)
Gebiet	-	-	9	(9)	-	-	4	(4)
Klasse	-	-	-	(0)	154	-	-	(154)
Gerade	-	26	-	(26)	-	-	-	(0)
Stück	-	-	-	(0)	7	-	-	(7)
Schnitt	-	-	-	(0)	-	45	-	(45)
Intervall	-	-	-	(0)	14	-	-	(14)
	(4)	(26)	(30)	(60)	(191)	(45)	(5)	(241)

I invite the reader to bear in mind all these figures—but especially the ones in (2), (4) and (6)—when reading the following step-by-step analysis of Dedekind's text.

4. Analysis of §§ 1-3.

The starting-point for Dedekind's work is a set which he takes as given:

(7) *Ich sehe die ganze Arithmetik als eine notwendige oder wenigstens natürliche Folge des einfachsten arithmetischen Aktes, des Zählens, an, und das Zählen selbst ist nichts anderes als die sukzessive Schöpfung der unendlichen Reihe der positiven ganzen Zahlen, in welcher jedes Individuum durch das unmittelbar vorhergehende definiert wird.*[9]

Dedekind talks of the set of positive integers as of an infinite series. In so choosing his words, he suggests to us the predicates that constitute the universe of discourse within which we can speak of an "object" different from any individual positive integer: this "object"—a set—is ordered (this is suggested by the noun *Reihe*) and it is infinite (*unendlich*), in contrast to non-infinite ordered sets, i.e. to finite series. An ontologist, however, would object here that, although it would be certainly absurd to say of any individual positive integer that it is ordered or that it is infinite, nevertheless it is perfectly possible and feasible to reduce set-talk by reducing the set-predicates involved in (7) to the two-place predicate 'to follow'—or its converse 'to be followed by'—so that to say that the set of positive integers is ordered and infinite is "really" to say that any positive integer is followed by one and only one positive integer different from itself.

Without entering into niceties (as e.g. that another two-place predicate, namely equality, is presupposed by the phrase *different from itself*), I would like to point out that the presence of a relational predicate like 'to follow' is a step towards that cohesion that characterizes a set. You can say that 'being followed by something or someone' is a property you have and not your follower, but I wonder whether this is a very interesting remark.[10]

On the other hand, the proposed reduction of order-cum-infinity to succession (plus appropriate quantifiers) constitutes at

it is not permissible within a given set or relatively to a given set. What I mean is that there is an increasing depth of predication beginning with order and infinity, which are predicates of a set of numbers, through commutativity (or associativity or distributivity), which are primarily predicates of operations, till permissibility, which is in a very interesting way a relational predicate that connects operations with sets.[12]

This is exactly the point where the words *System* and *Körper* appear:

(9) *Diese Beschränktheit in der Ausführbarkeit der indirekten Operationen ist jedesmal die eigentliche Ursache eines neuen Schöpfungsaktes geworden; so sind die negativen und gebrochenen Zahlen durch den menschlichen Geist erschaffen, und es ist in dem System aller rationalen Zahlen ein Instrument von unendlich viel grösserer Vollkommenheit gewonnen. Dieses System, welches ich mit R bezeichnen will, besitzt vor allen Dingen eine Vollständigkeit und Abgeschlossenheit, welche ich an einem anderen Orte als Merkmal eines Zahlkörpers bezeichnet habe, und welche darin besteht, dass die vier Grundoperationen mit je zwei Individuen in R stets ausführbar sind, d.h. dass das Resultat derselben stets wieder ein bestimmtes Individuum in R ist, wenn man den einzigen Fall der Division durch die Zahl Null ausnimmt.*[13]

Up to now we had a set of numbers (the set of positive integers) welded together by order and infinity—but now we have got a set of numbers (the set of rational numbers) that is welded together by much more interesting properties. It could even be affirmed that the original set became extended to a bigger set. It is also the transition from an ordered structure to an algebraic one, to use Bourbaki's phrase; and I suspect that the word *Körper* was chosen by Dedekind because of the suggestion of connectedness that this word has. Anyway, ontological reductionism ignores all this. For them, there aren't any sets, and a fortiori there aren't any series, finite or infinite or trigonometrical, and there aren't any systems, or any groups, rings or fields—there are only good old "individuals". But no; an ontologist would probably comment here that it was never his intention to recommend mathematicians to alter their languages. Well, it had better not, because no mathematician would listen to them if they tried to issue any such recommendation. But the point is rather, which way to ask questions has got the greater chances of improving our understanding of mathematical theory.

On the other hand, Dedekind is not interested here in algebra, but in analysis. And so he continues:

(10) *Für unseren nächsten Zweck ist aber noch wichtiger eine andere Eigenschaft des Systems R, welche man dahin aussprechen kann, dass das System R ein wohlgeordnetes, nach zwei entgegengesetzten Seiten hin unendliches Gebiet von einer Dimension bildet.*[14]

Observe the switch from *System* to *Gebiet* that is going to reappear so often afterwards; it is, as always, connected with the occurrence of "setty" predicates.

From the point of view of analysis the transition from the set of positive integers to the set of rational numbers is not so much an extension of operation permissibility as it is an extension of infinity, as we shall see. We could also say that a new kind of thinking begins here—topological thinking, to borrow again from Bourbaki—and with it a comparison between the set of rational numbers and the set of points in a straight line. (Let us not forget that set-theory has had, from its very inception, a great interest in comparing numbers and points in space.) Both sets seem to share three properties, formulated in three laws. The first law is harmless enough and does not really introduce anything new—not even with regard to the set of positive integers. I reproduce here for comparison the formulations of the law for both sets (the variables a,b,c range over the set of rational numbers and the variables p,q,r over the set of points in a straight line):

(11) a. *Ist $a > b$, und $b > c$, so ist $a > c$. Wir wollen jedesmal, wenn a,c zwei verschiedene (oder ungleiche) Zahlen sind, und wenn b grösser als die eine, kleiner als die andere ist, ohne Scheu vor dem Anklang an geometrische Vorstellungen dies kurz so ausdrücken: b liegt zwischen den beiden Zahlen a,c.*[15]

b. *Liegt p rechts von q, und q wieder rechts von r, so liegt auch p rechts von r; und man sagt, dass q zwischen den Punkten p und r liegt.*[16]

As I said before, this property applies also to the set of positive integers (to remark is only that it applies to three numbers). However, the second law introduces an infinity that was not in the set of positive integers:

(12) a. *Sind a,c zwei verschiedene Zahlen, so gibt es immer unendlich viele verschiedene Zahlen b, welche zwischen a,c liegen.*[17]

b. *Sind p,r zwei verschiedene Punkte, so gibt es immer unendlich viele Punkte q, welche zwischen p und r liegen.*[18]

The quantifier phrase *unendlich viele verschiedene Zahlen b* (or *unendlich viele verschiedene Punkte q*) obviates the use of set-talk, but the reader will perceive that such an expression implies a "setty" predicate—namely, infinity—and in that sense implies a set. We have here indeed a further step towards reification of sets. (I would also say that the other quantifier phrases, namely *zwei verschiedene Zahlen* and *zwei verschiedene Punkte*, suggest also something that is much more than two individual numbers or two individual points, namely a pair of numbers and a pair of points.)

The third law, finally, is in both cases formulated by using, not only our old acquaintance *System*, but also the new set-word *Klasse* (note that the word *Gerade* is employed as a set-word, and so is *Stück*):

(13) a. *Ist a eine bestimmte Zahl, so zerfallen alle Zahlen des Systems R in zwei Klassen, A_1 und A_2, deren jede unendlich viele Individuen enthält; die erste Klasse A_1 umfasst alle Zahlen a_1, welche $< a$ sind, die zweite Klasse A_2 umfasst alle Zahlen a_2, welche $> a$ sind; die Zahl a selbst kann nach Belieben der ersten oder der zweiten Klasse zugeteilt werden, und sie ist dann entsprechend die grösste Zahl der ersten oder die kleinste Zahl der zweiten Klasse. In jedem Falle ist die Zerlegung des Systems R in die beiden Klassen A_1, A_2 von der Art, dass jede Zahl der ersten Klasse A_1 kleiner als jede Zahl der zweiten Klasse A_2 ist.*[19]

b. *Ist p ein bestimmter Punkt in [einer geraden Linie] L, so zerfallen alle Punkte in L in zwei Klassen, P_1, P_2, deren jede unendlich viele Individuen enthält; die erste Klasse P_1 umfasst alle die Punkte p_1, welche links von p liegen, und die zweite Klasse P_2 umfasst alle die Punkte p_2, welche rechts von p liegen; der Punkt p selbst kann nach Belieben der ersten oder der zweiten Klasse zugeteilt werden. In jedem Falle ist die Zerlegung der Geraden L in die beiden Klassen oder Stücke P_1, P_2 von der Art, dass jeder Punkt der ersten Klasse P_1 links von jedem Punkte der zweiten Klasse P_2 liegt.*[20]

Now this is set-talk for you! Let us try and reduce it to the image of the ontologist. After introducing quantifiers where needed, we could reduce the first two sentences of (13a) to something like: *For any rational number a, there are infinitely many rational numbers a_1, a_2 such that $a_1 < a < a_2$.* But how could we express the rest? If we should say something like *and a could be considered either the greatest number a_1 or the smallest number a_2,* that would amount to a contradiction—saying once that *Any number a_1 is lesser than a* and again that *a equals some number a_1*, etc. I don't seem very good at ontology, but I shall try my hand again: *For any rational number a, there are infinitely many rational numbers a_1 and a_2 such that either $a_1 \leq a < a_2$* or $a_1 < a \leq a_2$. That seems to do the job all right; but what exactly have we done? This is a good place to remember that, although common nouns do not denote classes, they are nevertheless classifying artifacts. When we say *die rationalen Zahlen*, we do certainly not "refer to" the set of rational numbers, but we do classify the rational numbers. So, when Dedekind opposes the capital letters A_1, A_2 to the small-case letters a_1, a_2, he is in a sense only making explicit what was only implicit in a common noun or an NP—the question is, Why did he want to do such a thing? Our reduction has eliminated the capital letters—has it perhaps eliminated something else? Well, as a matter of fact it has: Dedekind was talking about an operation, namely dividing or partitioning a given set, which disappeared after the reduction. Should Dedekind have thought (and talked) like our ontological reductionist, then he would never have found his most interesting concept: the concept of a "cut" (*Schnitt*). Indeed, you will see that the quantifier phrase *infinitely many numbers a_1, a_2* doesn't imply here, as in (12a) a pair of numbers or even infinitely many pairs of numbers, but actually pairs of classes, i.e. not only sets, but sets of sets: these are the "cuts", as we shall see.

Another interesting point here is the use of variables. There is no linguistic theory of variables; variables are, however, linguistic "objects"; therefore, we have no theory of the variables at all. Such is the truth, despite all formal calculi and all logic textbooks. On the other hand, there is promise of a theory forthcoming from functional linguistics, especially from the research programme called UNITYP. According to Iturrioz (1986), variables are in many respects similar to numeral classifiers. This similarity would explain the behaviour of the variables A_1, A_2 in (13a); but, as all this is very new stuff, I won't go into it here.

A last observation: the appearance of the word *Stück* in (13b) is probably due to the fact that a straight line is conceived as a "mass", a continuous whole from which we extract a part. This motivates the use of a typical "measuring word" like *Stück*.[21] You can see here how linguistics and mathematics touch one another.

Although the third law contains a new infinity (what I call, after Dedekind, two-sided infinity), neither it nor the infinity talked of in the second law constitute yet continuity. Dedekind's § 3 opens with a geometrical consideration that leads to the conclusion that one of the two sets compared is larger than the other:

(14) *Die Gerade L ist unendlich viel reicher an Punktindividuen, als das Gebiet R der rationalen Zahlen an Zahlindividuen.*[22]

And so the task for the mathematician is clear:

(15) *Will man nun, was doch der Wunsch ist, alle Erscheinungen in der Geraden auch arithmetisch verfolgen, so reichen dazu die rationalen Zahlen nicht aus, und es wird daher unumgänglich notwendig, das Instrument R, welches durch die Schöpfung der rationalen Zahlen konstruiert war, wesentlich zu verfeinern durch eine Schöpfung von neuen Zahlen der Art, dass das Gebiet der Zahlen dieselbe Vollständigkeit oder, wie wir gleich sagen wollen, dieselbe Stetigkeit gewinnt, wie die gerade Linie.*[23]

(16) *Die obige Vergleichung des Gebietes R der rationalen Zahlen mit einer Geraden hat zu der Erkenntnis der Lückenhaftigkeit, Unvollständigkeit oder Unstetigkeit des ersteren geführt, während wir der Geraden Vollständigkeit, Lückenlosigkeit oder Stetigkeit zuschreiben. Worin besteht denn nun eigentlich diese Stetigkeit? In der Beantwortung dieser Frage muss alles enthalten sein, und nur durch sie wird man eine wissenschaftliche Grundlage für die Untersuchung aller stetigen Gebiete gewinnen.*[24]

I have quoted Dedekind at large not only to make clear his train of thought, but also to illustrate the differences between the words *System* and *Gebiet*. The comparison is announced in the title of § 2 as one between the rational numbers and the points in a straight line, but the talk is not of continuity, and not even of infinity, since this predicate, as we saw, was reduced, by Dedekind himself, to the quantifying expression *infinitely many*.

In § 3, however, Dedekind wants to talk explicitly of continuity, and he only uses the word *Gebiet*: the comparison has become one between the domain of the rational numbers and the straight line—a comparison between sets. Observe that in the only place where Dedekind could have talked of "the system R", namely in (15), he speaks rather of "the instrument R": we shall have occasion to affirm that *System* is a strongly operational word, but not totally operational, both similar to an instrument and to a set; thus, Dedekind prefers the more univocal word. Observe also the use of *Gebiet* as the head noun of a plural quantified phrase: it constitutes a proof of the greater individualization of that word as against *System*.[25]

Now, Dedekind had to think hard to find the definition:

(17) *Es kommt darauf an, ein präzises Merkmal der Stetigkeit anzugeben, welches als Basis für wirkliche Deduktionen gebraucht werden kann. Lange Zeit habe ich vergeblich darüber nachgedacht, aber endlich fand ich, was ich suchte. Dieser Fund wird von verschiedenen Personen vielleicht verschieden beurteilt werden, doch glaube ich, dass die meisten seinen Inhalt sehr trivial finden werden. Er besteht im folgenden. Im vorigen Paragraphen* [§ 2, cf. (13b)] *ist darauf aufmerksam gemacht, dass jeder Punkt p der Geraden eine Zerlegung derselben in zwei Stücke von der Art hervorbringt, dass jeder Punkt des einen Stückes links von jedem Punkte des anderen liegt. Ich finde nun das Wesen der Stetigkeit in der Umkehrung, also in dem folgenden Prinzip:*

"Zerfallen alle Punkte der Geraden in zwei Klassen von der Art, dass jeder Punkt der ersten Klasse links von jedem Punkte der zweiten Klasse liegt, so existiert ein und nur ein Punkt, welcher diese Einteilung aller Punkte in zwei Klassen, diese Zerschneidung der Geraden in zwei Stücke hervorbringt."[26]

As far as I can see this characterization—not yet a definition of continuity in general, but a characterization of the continuity of a straight line contains set-talk in an irreducible manner.[27] Because of that, I would dare to affirm that the formulation of (13a) was—though reducible—absolutely essential for Dedekind's thinking, because only through it was he enabled to see lawful connections that had escaped all great mathematicians who preceeded him. Moreover, it permitted him to establish that the sentence enclosed by double inverted commas in (17) is an axiom, i.e. space cannot be proved to be continuous, as against that extension of the set of rational numbers that we call the set of real numbers, for which it can be proved and was actually proved by Dedekind:

(18) *Hat überhaupt der Raum eine reale Existenz, so braucht er doch nicht notwendig stetig zu sein; unzählige seiner Eigenschaften würden dieselben bleiben, wenn er auch unstetig wäre. Und wüssten wir gewiss, dass der Raum unstetig wäre, so könnte uns doch wieder nichts hindern, falls es uns beliebte, ihn durch Ausfüllung seiner Lücken in Gedanken zu einem stetigen zu machen; diese Ausfüllung würde aber in einer Schöpfung von neuen Punktindividuen bestehen und dem obigen Prinzip gemäss auszuführen sein.*[28]

Such a creation of individuals is now the task: Dedekind must show how the set of rational numbers can be extended, so as to get a continuous set.

5. *Analysis of §§ 4-7.*

As I suggested before, continuity presupposes other properties, particularly the ones expressed in the three laws listed above as (11), (12) and (13). Thus, in order for Dedekind to prove that the set of real numbers is continuous, he must prove that it, too, has these properties. But again, to prove these properties, he must show what sort of "objects" are these real numbers. Dedekind's most important tool is the concept of a "cut", i.e. the word *Schnitt*, which he introduces in the following way:

(19) *Ist irgendeine Einteilung des Systems R in zwei Klassen A_1, A_2 gegeben, welche nur die charakteristische Eigenschaft besitzt, dass jede Zahl a_1 in A_1 kleiner ist als jede Zahl a_2 in A_2, so wollen wir der Kürze halber eine solche Einteilung einen Schnitt nennen und mit (A_1, A_2) bezeichnen.*[29]

As the complex sign "(A_1, A_2)" suggests, a cut is really a set of sets—a fact that has much to do with the irreducibility of set-talk in Dedekind's characterization of continuity. But we shall come to that in a moment.

Real numbers are cuts, although Dedekind hesitates to say as much. He prefers rather to say that a real number "produces" a cut. That is also the way he spoke about rational numbers: any rational number "produces" a cut. Now many cuts are "produced" by rational numbers, but not all; to be able to speak of all cuts, we need to introduce irrational numbers. Then we shall be able to say that any cut is "produced" either by a rational or by an irrational number, i.e. by a real number. It is at this point where the operation of partition ("produced" by a rational or irrational number) becomes reified into a cut, and the abstract noun *Schnitt* is taken from the technique of ABSTRACTION over to the technique of COLLECTION, to use the terms of UNITYP.[30] This will become clearer yet. But the consequence is very important: it will be nothing else than a redefinition of the very concept of number, i.e. a new way to apprehend numbers as "objects".

After giving an example that illustrates that there are infinitely many cuts that are not "produced" by rational numbers, Dedekind concludes:

(20) *In dieser Eigenschaft, dass nicht alle Schnitte durch rationale Zahlen hervorgebracht werden, besteht die Unvollständigkeit oder Unstetigkeit des Gebietes R aller rationalen Zahlen.*[31]

The next task is for Dedekind to prove that the set of real (i.e. rational and irrational) numbers is ordered, i.e. that for any two different real numbers that "produce" a cut, one of them follows the other. You could as well say that it is the cuts themselves that follow each other; as a matter of fact, Dedekind proceeds to compare cuts; only by means of such a comparison can he prove that the set of real numbers (the set of cuts) is ordered. Now, cuts are much more involved "objects" than sets of numbers or points: when you compare such a set with another, you put their elements (numbers with numbers or numbers with points) in relation to each other—but when you compare cuts, it is the "classes" (sets of numbers) which you put in relation to each other:

(21) *Um eine Grundlage für die Anordnung aller reellen, d.h. aller rationalen und irrationalen Zahlen zu gewinnen, müssen wir zunächst die Beziehungen zwischen irgend zwei Schnitten (A_1, A_2) und (B_1, B_2) untersuchen, welche durch irgend zwei Zahlen α und β hervorgebracht werden. Offenbar ist ein Schnitt (A_1, A_2) schon vollständig gegeben, wenn eine der beiden Klassen, z.B. die erste A_1, bekannt ist, weil die zweite A_2 aus allen nicht in A_1 enthaltenen rationalen Zahlen besteht, und die charakteristische Eigenschaft einer solchen ersten Klasse A_1 liegt darin, dass sie, wenn die Zahl a_1 in ihr enthalten ist, auch alle kleineren Zahlen als a_1 enthält.*[32]

As you can see, cuts are, as a matter of fact, sets of classes. Although it would be too long to analyse the proof of the ordering of the set of real numbers, I would like at least to point out the method of it from the point of view of the linguistic dimension of APPREHENSION.[33] Dedekind begins by comparing cuts, which are sets of classes; this leads him to compare the classes themselves (first step); but classes are sets of numbers; so he compares numbers (second step). Now, what are these "numbers"? Well, they are apprehended by means of Latin small-case letters. That means linguistically that they are not cuts. But Dedekind says once and again that e.g. $\beta = b_1$, or things like that. And he goes all the way up again to the cuts; and he says e.g.

(22) *In diesem Falle nennen wir die diesen beiden wesentlich verschiedenen Schnitten (A_1, A_2) und (B_1, B_2) entsprechenden Zahlen α und β ebenfalls verschieden voneinander.*[34]

It is as though Dedekind wouldn't quite accept that the Greek small-case letters "refer to" cuts. And in a sense this is quite right, because they don't apprehend cuts in the same way as the complex signs. It happens here something very similar to what happens in a Montague translation: individuals have in Montague semantics two faces—one face is the face of the variable ("the bindable objectual variable" of Quine's[35]) and the other face is the face of the second-order predicate. An individual is in a sense a set of properties, namely the set of properties that apply to that individual. Now, all this sounds absurd. An individual is an individual and not a set. This is best categorial thinking, the daily bread of ontologists. But when you begin to think non-categorially, but operationally, things begin to look better. The point is that in logic you begin with a pre-conception of an individual, a pre-conception that lasts as long as you don't look too close into the variables; when questions get more complicated, you need to rethink your individuals, and you redefine them as sets of properties. "But what are individuals in reality", asks the ontologist. This question hasn't any sense to me. The question is rather, "How do we apprehend individuals?" A very similar thing happens in Dedekind's text. He began with good old natural numbers (positive integers). They looked quite nicely ordered. But when he began to look more closely, they began to look less and less as independent beings in a row, and more and more like, say, "inflexions" of that row, i.e. like cuts. This is, in a nutshell, Dedekind's progress.

Now, after proving that the set of real numbers is ordered, Dedekind sets forth four theorems. The first three correspond quite exactly to the ones in (11), (12) and (13), so I won't go into them. But the fourth is the continuity theorem: it is a characterization of the set of real numbers in the fashion of that contained in (17). The proof of the theorem is a gem of simplicity and is, of course, accomplished by talking of cuts. From a linguistic point of view, there is an interesting observation to be made: in § 5, where these theorems are set forth, there are no less than five switchings from the word *Gebiet* to the word *System* and back. I cannot think of better evidence for my interpretation that we have here two very different ways of apprehension of one and the same set, in this case the set of real numbers. The section bears the title *Stetigkeit des Gebietes der reellen Zahlen*, but it begins with the words *Zufolge der eben festgesetzten Unterscheidungen bildet nun das System* ***R*** *aller reellen Zahlen...* That is the first switch. But the sentence ends like this: ... *ein wohlgeordnetes Gebiet von einer Dimension.* That is the second switch. After that come the theorems; now, since the first two don't contain any set-talk, there is no need for a switch; but we get it as soon as the third law is set forth: *Ist α eine bestimmte Zahl, so zerfallen alle Zahlen des Systems* ***R*** *in zwei Klassen...* That is the third switch. Now, as I said before, to these three laws a fourth is added: *Ausser diesen Eigenschaften besitzt aber das Gebiet* ***R*** *auch Stetigkeit.* That is the fourth switch. But in the formulation of the property or law the predicate of continuity doesn't appear, so we come back a last time: *Zerfällt das System* ***R*** *aller reellen Zahlen in zwei Klassen...* That is the fifth switch.[36] I ask: Why should Dedekind go so much out of his way to substitute expressions? Why shouldn't he use one word? Well, he uses two—nay, he uses ten words, and he makes all sorts of NPs with them, and he uses letters, and so on. You see that ontology has nothing to say about this. But, if we don't understand these facts, how can we hope to understand what Dedekind has done?

So the set of real numbers is continuous. But, can we calculate with real numbers as we can with natural and rational numbers? This is quite another question. To prove that we can, Dedekind uses cuts; as a matter of fact, operations with real numbers are operations with cuts. Dedekind shows how addition with cuts looks like, and says that the same can be done with all other operations, only—

(23) *Die Weitläufigkeiten, welche bei den Definitionen der komplizierteren Operationen zu befürchten sind, liegen teils in der Natur der Sache, zum grössten Teil aber lassen sie sich vermeiden. Sehr nützlich ist in dieser Beziehung*

der Begriff eines Intervalls, d.h. eines Systems A von rationalen Zahlen, welches folgende charakteristische Eigenschaft besitzt: sind a und a' Zahlen des Systems A, so sind auch alle zwischen a und a' liegenden rationalen Zahlen in A enthalten.[37]

This passage offers several points of interest. One of them is that here for the first time the word *System* is accompanied by a variable. Another is that, at least at this point, Dedekind becomes fully aware that the introduction of a set-word, in this case the word *Intervall*, helps thinking. It would be enough for me if my ontological readers should admit that set-talk—and not set-talk in general, but each particular sort, e.g. talk of intervals—really helps thinking. Indeed, a member of UNITYP has been able to prove that reification is necessary: in his case, namely abstract nouns, it was the reification of propositional contents, in our case here it is a different kind of reification.[38] Now, ontologists are prone to despise such reifications and to imply that there is no interest in studying the mental operations that underlie them. I would like to insist here that such operations are not simply the strange antics of undisciplined minds, but the very stuff which all our thinking is made of. Here lies also, I believe, the explanation of the use of the word *System* with a variable: it is the most operational of Dedekind's set-words, a sort of half-breed between the elements of a set and a set of these elements. And Dedekind seems to need such a half-breed to bring about the reifications he needs. Thus he can, inter alia, achieve great clarity of thought with its help, namely when it comes to the relations between intervals within the domain of real numbers:

(24) *Das ganze Gebiet R zerfällt in drei Stücke, A_1, A, A_2, und es treten zwei vollständig bestimmte rationale oder irrationale Zahlen α_1, α_2 auf, welche bzw. die untere und obere (oder die kleinere und grössere) Grenze des Intervalls A genannt werden können; die untere Grenze α_1 ist durch den Schnitt bestimmt, bei welchem die erste Klasse durch das System A_1 gebildet wird, und die obere Grenze α_2 durch*

den Schnitt, bei welchem A_2 die zweite Klasse bildet.[39]

In all prior cases of partitions, the partitioned set was called a *System* and the subsets that resulted from the partition were called *Klassen*; but here the description becomes more complicated because of the interval A. Thus, the word *System* yields to the hierarchically superior word *Gebiet*, and thinking can begin to apprehend each object involved: the domain has three parts, the systems A_1, A_2, and the interval A between those systems; this interval functions now like a rational number in that it produces two—two classes of course, which are related to the systems mentioned; in fact, they are (extensionally) identical with them.

Now, after defining the algebraic operations (a task not fulfilled, but only indicated here), there comes the much more intricate problem of proving that these operations possess, within the set of real numbers, the same important properties that they possess within the set of rational numbers (i.e. commutativity and so on). Dedekind says that for this task not even the word *Intervall* would be enough—he gives an illustration of a theorem where only intervals are talked of and comments on it:

(25) *Die abschreckende Schwerfälligkeit aber, welche dem Ausspruche eines solchen Satzes anklebt, überzeugt uns, dass hier etwas geschehen muss, um der Sprache zu Hilfe zu kommen; dies wird in der Tat auf die vollkommenste Weise erreicht, wenn man die Begriffe der veränderlichen Grössen, der Funktionen, der Grenzwerte einführt, und zwar wird es das Zweckmässigste sein, schon die Definitionen der einfachsten arithmetischen Operationen auf diese Begriffe zu gründen, was hier jedoch nicht weiter ausgeführt werden kann.*[40]

As a good mathematician, he is not interested in such a bizarre question as whether e.g. functions exist or not. He needs them *to think* and—even more important—he is able to apprehend them linguistically.

Thus we have come to the culmination of this little treatise: Dedekind's last task is to show how his definition of continuity

can be used to prove the most fundamental theorems of calculus (infinitesimal analysis, as he says). Anybody that examines carefully the proofs in § 7 will be able to see how firmly entrenched in Dedekind's thought the new language is: "classes" are not spoken of any more, instead of them the "domain" is divided into two "systems" by a real number that "belongs" to a "system" and "produces" a "cut", etc. The proofs are simply beautiful in their clarity and simplicity. They are, I would like to remind the reader, the first proofs ever given for the most fundamental theorems of the most important mathematical tool ever devised by human beings.

6. *On method*

I would like to draw a moral from the above story. There are some peculiar features of real set-talk (I mean the one you find in the writings of mathematicians unworried by philosophical troubles) that philosophers have consistently ignored in their discussions. In my analysis I have referred to three of them:

A. Set-talk is seldom—or never—homogeneous, i.e. mathematicians are not in the habit of jusing just one word to talk of sets, but they indulge in a wide variety of set-words, talking alternatively of *sets*, *classes*, *aggregates*, *collections*, *domains*, *families*, *systems*, and what-not; and then again of *fields*, *rings* and *groups*; and then again of *successions*, *series* and *n-tuples*. Methodologically speaking, an analysis of set-talk should start here—making a list of all the set-words a given theory needs or a given mathematicians employs. There are lots of questions to be asked, e.g. from the semantical viewpoint: Where do set-words come from? Which are the preferred metaphors (in the sense of Lakoff & Johnson 1980? How do they build together a semantic field?

B. The appearance of set-talk goes hand in hand with the introduction into the mathematical discourse of certain predicates. These predicates constitute a continuum: some of them apply only to the elements, others both to them and to the set —and this under more or less severe restrictions—and still others only to the set. Thus, you can say *the real numbers are* the same time a reduction of Dedekind's talk of the operation or act of counting. But besides counting, there are other operations, and so Dedekind continues:

(8) *Die Kette dieser Zahlen bildet an sich schon ein überaus nützliches Hilfsmittel für den menschlichen Geist, und sie bietet einen unerschöpflichen Reichtum an merkwürdigen Gesetzen dar, zu welchen man durch die Einführung der vier arithmetischen Grundoperationen gelangt. Die Addition ist die Zusammenfassung einer beliebigen Wiederholung des obigen einfachsten Aktes zu einem einzigen Akte, und aus ihr entspringt auf dieselbe Weise die Multiplikation. Während diese beiden Operationen stets ausführbar sind, zeigen die umgekehrten Operationen, die Subtraktion und Division, nur eine beschränkte Zulässigkeit.*[11]

In the first sentence, Dedekind alludes to laws such as *Addition (of positive integers) is commutative*, where the predicate *is commutative* applies to an operation, but can be reduced like this: *For any two positive integers a,b it is the case that $a+b=b+a$*. Thus, if addition can be reduced to counting, as Dedekind himself suggests in the second sentence, we would come again to our predicate of succession (and that of equality) which applies to individual numbers. It may be. I should like to remark only that this new reduction contains a reference to two numbers, and so, in spite of ontologists, it is much "settier" than the preceding reduction. (As a matter of fact, most mathematicians like to formulate the above law starting with *For any pair of positive integers a,b ...*; and what is a "pair" but a set?)

Now, in his third sentence Dedekind introduces explicitly the predicate *ausführbar* (or *zulässig*, 'permissible'), which at first sight is like *commutative* in that it applies to operations and can be reduced like this (for subtraction): *It is not the case that for any two positive integers a,b there is a third positive integer c such that $a-b=c$*. I think that the peculiar interest of a predicate like *permissible* is that, in contrast to *commutative*, it is markedly a two-place predicate: you don't simply say of subtraction that it is not permissible, you say rather that

infinite, but not **the real numbers are continuous*. Lots of things can be done here, too. I shall come to one of them in the next section.

C. Set-words are in general head nouns of more or less complex NPs. Nay, even before that you have the possibility of e.g. composition. In general, I think that the best methodology would be here to distinguish with UNITYP three dimensions: DESCRIPTIVITY, APPREHENSION and DETERMINATION.[41] In this paper, I have limited myself to some remarks relative to APPREHENSION, e.g. the appearance of variables, the question of extraction, the levels of generalization, etc. But a more complete analysis should go very much beyond that. Anyway, I would like to remind the reader that set-NPs can attain a very great complexity, let us say from *all sets* through *the set of all real numbers* to *every countable dense linearly ordered set without points*. I have even found in mathematical books set-NPs that were half a page long!

My claim is simply that, without a linguistic analysis of these three—and maybe other—features of real set-talk, any philosophical treatment can only be incomplete and unsound. Why, it may even verge on the ridiculous, as is the case of Black (1970), where even some didactical advices (in a philosophical journal, too!) are given as to what one should tell children and what not. So for instance, you shouldn't talk of sets to them, because there aren't any; you should only use plural NPs, and so on. (His "linguistic" observations, as for example that NPs like *the Cabinet* are "easily" reduced to NPs like *the members of the Cabinet*, are simply beneath contempt.) But above all: It is not at all clear what all these precepts and injunctions contribute to a better understanding of set-talk.

7. Beyond set-talk

I would like finally to point out a possible application of this necessarily too brief case study to the proper field of linguistics. What I have, quite informally, called set-words are, from a linguistic point of view, collective nouns. Now collective nouns have so far been something of a step-child within linguistic theory—at least, I would dare to say, till Kuhn's papers, written in the framework of UNITYP.[42] In particular, I find his analysis of the "meaning" of collective expressions into three "components" very exciting, although his characterization of the "element quality" and the "unifying quality" are too metaphysical for my taste. Thus, when he says that the "element quality" is "a quality that every element of the collection has to have [namely, in order to belong to that collection]", one wonders what he is talking about—of language or of reality? And, if the "unifying quality" is "that property on the basis of which objects are brought together into a collection", is that talk still linguistics? (Cf. Kuhn 1982a:91, 1982b:56.) On the other hand, I am not sure that I understand the following sentence: "In a collective expression, e.g. German *eine Herde Schafe*, the word *Herde* indicates the property on whose basis the set of sheep designated by the expression is apprehended as a collection, i.e. the unifying quality; and *Schafe* names the element quality, i.e. the property which must apply to any object that wants to be an element of the collection, and it must apply to that object independently of the collection." (Kuhn 1982b:57.) Besides the somewhat vague talk of "indicating" and "naming", I don't find this very useful, either for sheep or for rational numbers. What is the element property of rational numbers? I suppose that they are rational numbers. But what is it to be a rational number? I think that can only be answered by analyzing the discourse on rational numbers. And I suggested above that it is at least doubtful that rational numbers can be defined in complete independence of at least other elements of the set. I don't see how we can have real progress here, if we don't try to develop the above proposal of a "continuum of predication". And such a development should have consequences for the general study of the syntax and semantics of collective expressions.

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FOOTNOTES

[1] I am depicting a general attitude without caring very much for fine differences. Typical examples of this attitude are Russell (1919:ch. XVII) and Black (1970), on all other counts very different philosophers indeed. And of course Quine is the undisputed master ontologist and maybe the only one who has really gone beyond programmatic sketches and superficial observations (albeit without doing real linguistic analysis), admitting even to an "irreducible core" of set-theory and therefore of set-talk (cf. Quine 1960:ch. VII, esp. §55; 1982:§46). Incidentally, I took the expression 'set-talk' from Black (1970), who didn't see, of course, its explosive potential ('set-talk' is not simply 'talk of sets', as the reader will see for himself).

[2] Cf. Iturrioz/Leal 1986. This position underlies the best work in Seiler/Lehmann eds. 1982 and Seiler/Stachowiak eds. 1982. It is not an accident that the by far ablest of ontologists, namely Quine, recognizes the connexion between "objecthood" and theory (and therefore language), and this not only in the case of sets or classes (where you could always think he was forced by the sheer fact of a certain "irreducible core", cf. note 1), but also in the for him vital case of "physical objects" (cf. Quine 1960:238).

[3] It wouldn't be difficult to prove that, besides the work by Quine (and that of the so-called "Erlanger Schule" lead by Paul Lorenzen), there has been no real advances forthcoming from ontologists. And even this work has been, as I think, too much dominated by a prescriptivist attitude that has made their few attempts at description and analysis of language much more unfruitful than they deserved to be.

[4] The need for a linguistic concept of "continua" is a sign of our times, but I think that nobody has seen more clearly than Seiler into the nature and capabilities of that concept. Also, and very important indeed, only Seiler and his co-workers of the Cologne UNITYP project have developed a complex theory within which continua play a non-isolated but systematic role side by side with many other theoretical concepts. Since all work that Seiler has either written or inspired is a variation on that theme, the only honest thing I can do here, by way of reference, is to invite the reader to consult the bibliographical section of this volume.

[5] Cf. note 1. The idea that you can only talk of "objects" within a given universe of discourse is found in the "Erlanger Schule", cf. e.g. Lorenzen 1974:198; but it was further elaborated and made useful for linguistics by Iturrioz (cf. 1985a:29ff.).

[6] For an elementary exposition see Meschkowski 1973:VI. It would be of extraordinary interest to test this idea by analyzing Bourbaki's set-talk through all their volumes.

[7] I enclose the metalinguistic predicate "refer to" within scare-quotes because one of the points of the present exercise is precisely to show that reference is not the whole of semantics. As Seiler and his co-workers have shown once and again, the whole issue of "apprehension of objects" is presupposed by and for reference, or more precisely by and for determination, because reference is only one aspect of a complex dimension. Cf. Seiler's contribution to this volume. By the way, I put also the noun "object" within scare-quotes, but in this case what I intend to suggest is that I am very far of being a "realist" in dispute with the wrong-headed "nominalists". From my point of view, both sides make exactly the same mistake, namely to believe in the primacy of ontology. "Object" is for me an operational concept: an object is nothing if it is not constituted by language (or anyway by the human mind).

[8] The word "extraction" stems from UNITYP, as does also "separation", cf. the contributions to Seiler/Lehmann eds. 1982. It is interesting to observe the following regularity: a set-word can be accompanied by a variable if and only if it "refers to" a set that is "extracted" from a given set.

[9] Dedekind 1872:5(§1).

[10] On the other hand, note that, although Dedekind does not speak about this, it is a property of the set of positive integers to have a first member; and if you want to reduce this property, you shall have to say something like "there is one and only one positive integer such that it does not follow any positive integer", thereby postulating that every positive integer has a certain relation to one "designated" positive integer—which is a step further yet in the said direction. Indeed, as everyone knows, the quite "setty" predicate of well-ordering is based upon this property.

[11] Dedekind 1872:5-6(§1).

[12] Of course, I know that commutativity can also be called a relational predicate in that e.g. addition is commutative also relatively to positive integers; but I think that the historical sequence of events corresponds to the proposed scale—and also that this sequence may very well reflect the conceptual sequence every mathematician has to live through. (As to the use of "commutative" as a predicate of sets, e.g. of rings, it is quite a different matter, namely a case of scope-narrowing metonymy.) Now, to fully understand the scale, one should undertand the connexion between sets and operations. The set of the positive integers is born from counting, as Dedekind quite rightly says, so in a sense operations are before sets, or even more drastically expressed: operations constitute sets. (That is Frege's position, cf. 1874:2.) But to go into it would imply a digression into abstract nouns among other things; so let it be a solitary remark that awaits development.

[13] Dedekind 1872:6(§1).

[14] Dedekind 1872:6(§1).

[15] Dedekind 1872:7(§1).

[16] Dedekind 1872:8(§2).

[17] Dedekind 1872:7(§1).

[18] Dedekind 1872:8(§2).

[19] Dedekind 1872:7(§1).

[20] Dedekind 1872:8(§2).

[21] Cf. Drossard 1982.

[22] Dedekind 1872:9(§3).

[23] Dedekind 1872:9(§3)

[24] Dedekind 1872:10(§3).

[25] Cf. Kuhn 1982a:87.

[26] Dedekind 1872:10(§3).

[27] Cf. note 1.

[28] Dedekind 1872:11(§3).

[29] Dedekind 1872:11(§4).

[30] Cf. Iturrioz 1982b.

[31] Dedekind 1872:13(§4).

[32] Dedekind 1872:13-14(§4).

[33] About APPREHENSION see Seiler/Lehmann eds. 1982, Seiler/Stachowiak eds. 1982, and Seiler (forthcoming).

[34] Dedekind 1872:15(§4).

[35] Quine 1977:272.

[36] All these quotations are taken from Dedekind 1872:§5. I have substituted the gothic letter of the original for a bold face italic (***R***).

[37] Dedekind 1872:18(§6).

[38] Cf. Iturrioz 1982a, 1982b, 1985a, 1985b.

[39] Dedekind 1872:19(§6).

[40] Dedekind 1872:19(§6).

[41] Cf. Iturrioz/Leal 1986.

[42] Cf. Kuhn 1982a, 1982b.

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